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BENDING OF A CONSTRAINED CIRCULAR BEAM RESTING ON AN ELASTIC FOUNDATION

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BENDING OF A CONSTRAINED CIRCULAR BEAM RESTING ON AN ELASTIC FOUNDATION

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The solution of the problem of a circular beam resting on an elastic foundation, loaded by concentrated symmetric forces and constrained in such a way that the sections where the vertical loads are applied cannot rotate, is given in explicit form.

Tables for the deflections, angles of twist, bending and twisting moments are presented.

Nomenclature

The following nomenclature is used in the paper:

M_x = bending moment

M_z = twisting moment

v = displacement of axis of beam in direction perpendicular to plane of beam

β = angle of twist

$1/R$ = initial curvature of center line of beam

I_x = moment of inertia of cross-section of beam with respect to x axis

C = torsional rigidity of beam

E = Young's modulus of beam

$y = \frac{v}{R}$

$\mu = C/EI_x$

$\alpha = EI_x/R$

P = concentrated force

k = elastic constant of foundation

2γ = angular distance between points of application of external forces

$\lambda = KR^4/EI_x$

$\gamma = R^2P/EI_x$

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Introduction

In previous papers [1], [2], [3][1], the problem of the deflections of a circular beam resting on an elastic foundation and loaded by symmetric or antisymmetric forces has been studied. The beam was then supposed to be free to rotate in correspondence with the sections where the loads were applied.

However, very frequently (as was pointed out by Professor R. M. Mindlin of Columbia University during the discussion which followed the presentation of paper No. [1]) the vertical concentrated loads are transmitted to the circular beam through structural members which are rigidly connected to the beam. These structural members will restrain the beam at the points of application of the concentrated loads from rotating about the center line of the beam (see Fig. 1).

The object of this paper is to analyze deflections of circular beams under such conditions and to express their deflections, angles of twist, and bending and twisting moments in explicit forms as functions of the externally applied forces P, the inertia characteristics EI_x of the beam, the elastic constant k of the foundation and the angular distance 2γ between the forces.

For particular values of the angles 2γ, namely: $2\gamma = 120^\circ$; 90°, 60°, 45° and 30°, Tables giving the values of the deflections, angles of twist and bending and twisting moments are presented.

The following assumptions will be made:

(a) The beam has constant inertia characteristics EI_x and its axis is a circle of radius R.

(b) At equidistant points, equal concentrated forces P are applied, the angular distance between the forces being indicated by 2γ (see Fig. 1).

(c) The sections where the loads are applied are not free to rotate.

(d) The foundation reacts following the Winkler-Zimmermann hypothesis, i.e., the reactive forces due to the foundation are proportional at every point to the deflection of the beam at that point [4], [5], [6].

Constrained Circular Beam Resting on an Elastic Foundation

The beam is referred to a system of rectangular coordinates OXYZ with the origin O at the centroid of the cross-section, the axes X and Y coinciding with the principal axes of inertia of the cross-section and the axis Z with the tangent to the center line at O, as shown in Fig. 2. If v is the displacement of the centroid O in the direction of the Y axis, β the angle of twist of the cross-section about the Z axis, M_x, M_z the moments acting on the cross-section at O about the X and Z axes, EI_x the flexural rigidity, C the torsional rigidity, 1/R the initial curvature of the center line of the bar, the following two equations will be verified [1], [3], [7], [8], [9]:

$$\left.\begin{aligned} EI_x\left(\frac{\beta}{R} - \frac{d^2 v}{ds^2}\right) &= M_x \\ C\,\Theta = C\left(\frac{d\beta}{ds} + \frac{1}{R}\frac{dv}{ds}\right) &= M_z \end{aligned}\right\} \qquad (1)$$

If k is the elastic constant of the foundation, P the external concentrated force, 2γ the angular distance between the points of application of the external forces (see Figs. 1 and 2):

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[1] Numbers in brackets refer to Bibliography at end of paper.

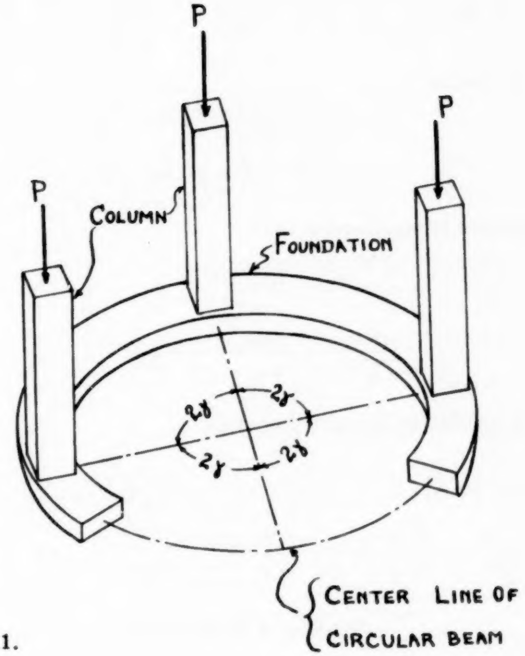
P
P
P
COLUMN
FOUNDATION
2γ
2γ
2γ
2γ
CENTER LINE OF
CIRCULAR BEAM

Fig. 1.

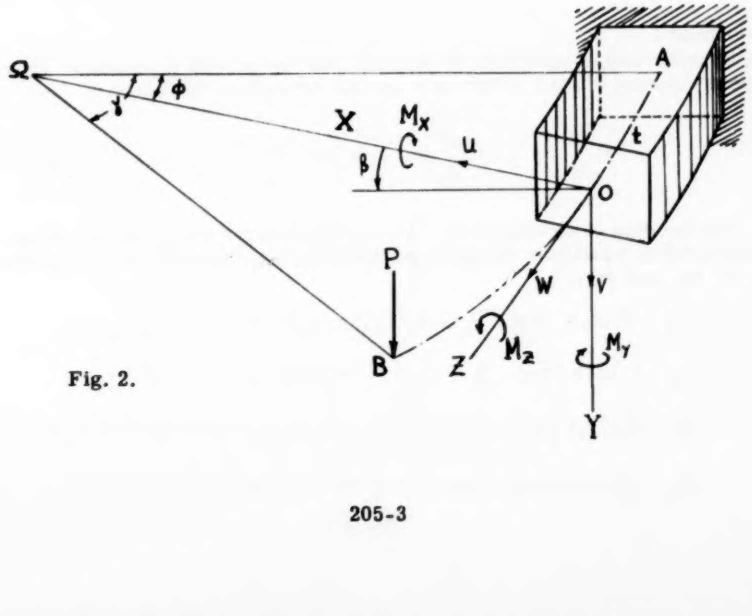
Ω
γ
φ
X
M_X
u
β
A
t
O
P
W
V
B
Z
M_Z
M_Y
Y

Fig. 2.

$$y = \frac{v}{R} ;\quad a = \frac{EI_x}{R} ;\quad \mu = \frac{C}{EI_x} ;\quad \nu = \frac{PR^2}{EI_x} ;\quad \lambda = \frac{KR^4}{EI_x}$$

equations (1) become:

$$\left.\begin{aligned} a\left(\beta - \frac{d^2 y}{d\varphi^2}\right) &= M_x \\ \mu a\left(\frac{d\beta}{d\varphi} + \frac{dy}{d\varphi}\right) &= M_z \end{aligned}\right\} \tag{2}$$

The conditions of equilibrium are:

$$\left.\begin{aligned} &\frac{d m_z}{d\varphi} - m_x = 0 \\ &\frac{d^2 m_x}{d\varphi^2} + \frac{d m_z}{d\varphi} - a\lambda y = 0 \end{aligned}\right\} \tag{3}$$

the boundary conditions being in this case:

$$\left.\begin{aligned} &\beta(\gamma) = \beta(-\gamma) = 0 ;\quad y(\gamma) = y(-\gamma) ;\quad \frac{dy(\gamma)}{d\varphi} = -\frac{dy(-\gamma)}{d\varphi} \\ &\frac{d^2 y(\gamma)}{d\varphi^2} = \frac{d^2 y(-\gamma)}{d\varphi^2} ;\quad \frac{d m_x(\gamma)}{d\varphi} - \frac{d m(-\gamma)}{d\varphi} = a\nu \end{aligned}\right\} \tag{4}$$

Solution of the Problem

By combining equations (2) and (3) the following expression for β in terms of y and its derivatives:

$$\beta = \frac{\mu}{1+\mu}\frac{d^4 y}{d\varphi^4} + \frac{1+2\mu}{1+\mu}\frac{d^2 y}{d\varphi^2} + \frac{\lambda\mu}{1+\mu}\frac{dy}{d\varphi} \tag{5}$$

is obtained.

By substituting equations (3) and (5) into the second of equations (2) the linear equation of sixth order in y and its derivatives is obtained:

$$\mu\frac{d^6 y}{d\varphi^6} + 2\mu\frac{d^4 y}{d\varphi^4} + \mu(1+\lambda)\frac{d^2 y}{d\varphi^2} - \lambda y = 0 \tag{6}$$

The solution of equations (6), (5) and (2) with the boundary conditions expressed by equations (4) gives the following expressions for the functions y, β, M_x and M_z:

$$y = A\cosh(m\varphi) + B_1\cosh(r\varphi)\cos(s\varphi) + B_2\sinh(r\varphi)\sin(s\varphi)$$

$$\beta = C\cosh(m\varphi) + D_1\cosh(r\varphi)\cos(s\varphi) + D_2\sinh(r\varphi)\sin(s\varphi)$$

$$M_x = \left[E\cosh(m\varphi) + F_1\cosh(r\varphi)\cos(s\varphi) + F_2\sinh(r\varphi)\sin(s\varphi)\right]a$$

$$M_z = \left[G\sinh(m\varphi) + H_1\sinh(r\varphi)\cos(s\varphi) + H_2\cosh(r\varphi)\sin(s\varphi)\right]a$$

where m, r, s are determined from the formulas:

$$m = \sqrt{\xi + \eta - \frac{2}{3}}$$

$$r = \frac{1}{2}\sqrt{\frac{2}{m}\sqrt{\frac{\lambda}{\mu}} - m^2 - 2}$$

$$s = \frac{1}{2}\sqrt{\frac{2}{m}\sqrt{\frac{\lambda}{\mu}} + m^2 + 2}$$

with:

$$\xi = \frac{1}{3}\sqrt[3]{9\lambda + 1 + \frac{27\lambda}{2\mu} + \sqrt{27\lambda(\lambda+1)^2 + \frac{9\lambda+1}{\mu} + \frac{27\lambda}{4\mu^2}}}$$

$$\eta = \frac{1}{3}\sqrt[3]{9\lambda + 1 + \frac{27\lambda}{2\mu} - \sqrt{27\lambda(\lambda+1)^2 + \frac{9\lambda+1}{\mu} + \frac{27\lambda}{4\mu^2}}}$$

The constants A............Y are calculated from the following formulas:

$$A = \frac{r(rP + sQ)}{2m\left[rR + sS - \frac{2rs}{1+\mu}\sqrt{\frac{\lambda}{\mu}}\,T\right]}$$

$$B_1 = -\frac{r\left[\mu m U \sinh(m\gamma) + \left(m^2 + \frac{\lambda}{m^2} - \mu\right) X \cosh(m\gamma)\right]}{2\mu m\left[rR + sS - \frac{2rs}{1+\mu}\sqrt{\frac{\lambda}{\mu}}\,T\right]}$$

$$B_2 = -\frac{r\left[\mu m V \sinh(m\gamma) - \left(m^2 + \frac{\lambda}{m^2} - \mu\right) Y \cosh(m\gamma)\right]}{2\mu m\left[rR + sS - \frac{2rs}{1+\mu}\sqrt{\frac{\lambda}{\mu}}\,T\right]}$$

where

$$P = \left[-(1+m^2)^2 - \frac{1}{m}\left(m^2 - \frac{1}{\mu}\right)\sqrt{\frac{\lambda}{\mu}}\right]\sin(s\gamma)\cos(s\gamma)$$

$$Q = \left[-(1+m^2)^2 + \frac{1}{m}\left(m^2 - \frac{1}{\mu}\right)\sqrt{\frac{\lambda}{\mu}}\right]\sinh(r\gamma)\cosh(r\gamma)$$

$$R = \left\{\lambda(1+m^2)\left(\frac{1}{\mu m^2} - 2\right) + \left[m(1+m^2) - \frac{\lambda}{m^3}\left(m^2 - \frac{1}{\mu}\right)\sqrt{\frac{\lambda}{\mu}}\right]\right\}\sinh(m\gamma)\sin(s\gamma)\cos(s\gamma)$$

$$S = \left\{\lambda(1+m^2)\left(\frac{1}{\mu m^2} - 2\right) - \left[m(1+m^2) - \frac{\lambda}{m^3}\left(m^2 - \frac{1}{\mu}\right)\sqrt{\frac{\lambda}{\mu}}\right]\right\}\sinh(m\gamma)\sinh(r\gamma)\cosh(r\gamma)$$

$$T = \frac{1}{m^2}(1+m^2)\left(m^2+\frac{\lambda}{m^2}-\mu\right)\cosh(m\gamma)\left[\sinh^2(r\gamma)\cos^2(s\gamma)+\cosh^2(r\gamma)\sin^2(s\gamma)\right]$$

$$U = \left[\frac{1}{2}\left(m^2+\frac{1}{\mu}\right)(m^2+2)+1\right]\sinh(r\gamma)\sin(s\gamma)+2rs\left(m^2-\frac{1}{\mu}\right)\cosh(r\gamma)\cos(s\gamma)$$

$$V = -\left[\frac{1}{2}\left(m^2+\frac{1}{\mu}\right)(m^2+2)+1\right]\cosh(r\gamma)\cos(s\gamma)+2rs\left(m^2-\frac{1}{\mu}\right)\sinh(r\gamma)\sin(s\gamma)$$

$$X = r\cosh(r\gamma)\sin(s\gamma)+s\sinh(r\gamma)\cos(s\gamma)$$

$$Y = r\sinh(r\gamma)\cos(s\gamma)-s\cosh(r\gamma)\sin(s\gamma)$$

$$E = \frac{A\mu}{1+\mu}\left(\frac{\lambda}{\mu m^2}-m^2-1\right)$$

$$F_1' = -\frac{\mu}{2(1+\mu)}(1+m^2)\left(m^2B_1+4rsB_2\right)$$

$$F_2' = -\frac{\mu}{2(1+\mu)}(1+m^2)\left(m^2B_2-4rsB_1\right)$$

$$G = \frac{E}{m}$$

$$H_1 = m\sqrt{\frac{\mu}{\lambda}}\left(rF_1'-sF_2'\right)$$

$$H_2 = m\sqrt{\frac{\mu}{\lambda}}\left(rF_2'+sF_1'\right)$$

Numerical Tables

The values of the functions $\frac{\beta}{v}$, $\frac{y}{v}$, $\frac{M_x}{av}$, $\frac{M_z}{av}$ are given in Tables 1, 2, 3, 4 and 5 for the following values of the parameters:

$$\lambda = 60,\ \mu = 0.7;\quad \lambda = 100,\ \mu = 0.7$$
$$\lambda = 60,\ \mu = 1.0;\quad \lambda = 100,\ \mu = 1.0$$

The numerical computations involved in the preparation of these Tables are very laborious. At first the authors had prepared Tables with four decimal figures. However, for practical use it is advisable to have Tables with a

greater accuracy, i.e., with a greater number of decimals, and these it is not practical to prepare without the use of proper computing machines. The Tables which are presented with six decimal figures were calculated by computers of the "Istituto per le Applicazioni del Calcolo" of the Italian "Consiglio Nazionale delle Ricerche" in Rome. The authors want to express their appreciation and thanks to the Director of the "Istituto per le Applicazione del Calcolo", Professor Mauro Picone, for the generous help received.

Table 1. Deflections, angles of twist, and bending and twisting moments for $2\gamma = 120^0$

	ϕ	$\frac{y}{V}$	$\frac{\beta}{V}$	$\frac{M_x}{aV}$	$\frac{M_z}{aV}$
	0°	.000765	.025509	-.027786	0
	10°	.001570	.024198	-.026544	-.004780
λ = 60	20°	.003902	.020025	-.021398	-.009044
	30°	.007469	.013813	-.008300	-.011791
μ = 0.7	40°	.011673	.006654	.018768	-.011130
	50°	.015458	.000806	.066487	-.004035
	60°	.017188	.000000	.140059	.013592
	0°	.000835	.023236	-.029301	0
	10°	.001629	.021999	-.027973	-.005039
λ = 60	20°	.003929	.018414	-.022594	-.009534
	30°	.007454	.012936	-.009176	-.012463
μ = 1.0	40°	.011614	.006579	.018209	-.011926
	50°	.015370	.001232	.066157	-.004908
	60°	.017092	.000000	.139811	.012671
	0°	-.000520	.019018	-.016618	0
	10°	.000021	.018112	-.017019	-.002927
λ = 100	20°	.001627	.015406	-.016567	-.005897
	30°	.004188	.011042	-.010423	-.008384
μ = 0.7	40°	.007367	.005655	.008925	-.008770
	50°	.010391	.000981	.050354	-.003982
	60°	.011843	.000000	.121438	.010543
	0°	-.000482	.017425	-.017854	0
	10°	.000054	.016615	-.018180	-.003138
λ = 100	20°	.001644	.014201	-.017521	-.006295
	30°	.004182	.010318	-.011092	-.008924
μ = 1.0	40°	.007337	.005515	.008543	-.009401
	50°	.010344	.001175	.050182	-.004660
	60°	.011791	.000000	.121342	.009844

Table 2. Deflections, angles of twist, and bending and twisting moments for $2\gamma = 90^0$

	ϕ	$\frac{y}{\gamma}$	$\frac{\beta}{\gamma}$	$\frac{M_x}{a\gamma}$	$\frac{M_z}{a\gamma}$
$\lambda = 60$, $\mu = 0.7$	0^0	.006335	.017598	-.043027	0
	10^0	.007237	.015783	-.036459	-.007130
	20^0	.009686	.010827	-.015351	-.011885
	30^0	.012885	.004640	.024001	-.011429
	40^0	.015440	.000163	.086021	-.002185
	45^0	.015869	.000000	.126474	.007040
$\lambda = 60$, $\mu = 1.0$	0^0	.006379	.015396	-.044266	0
	10^0	.007267	.013650	-.037641	-.007343
	20^0	.009660	.009670	-.016389	-.012293
	30^0	.012837	.004292	.023126	-.012003
	40^0	.015367	.000305	.085252	-.002901
	45^0	.015702	.000000	.125720	.006258
$\lambda = 100$, $\mu = 0.7$	0^0	.002622	.015340	-.035777	0
	10^0	.003385	.013815	-.031054	-.005974
	20^0	.005479	.009650	-.014760	-.010174
	30^0	.008269	.004207	.018815	-.010124
	40^0	.010556	.000177	.076825	-.002161
	45^0	.010948	.000000	.116707	.006209
$\lambda = 100$, $\mu = 1.0$	0^0	.002659	.013451	-.036951	0
	10^0	.003411	.012148	-.032157	-.006174
	20^0	.005478	.008579	-.015681	-.010552
	30^0	.008238	.003885	.018099	-.010645
	40^0	.010504	.000296	.076243	-.002813
	45^0	.010894	.000000	.116143	.005527

Table 3. **Deflections, angles of twist, and bending and twisting moments for $2\gamma = 60^\circ$**

	ϕ	$\frac{y}{\gamma}$	$\frac{\beta}{\gamma}$	$\frac{M_x}{a\gamma}$	$\frac{M_z}{a\gamma}$
$\lambda = 60$, $\mu = 0.7$	0°	.014517	.006523	−.039072	0
	5°	.014659	.006242	−.035562	−.003305
	10°	.015174	.005168	−.025103	−.006004
	15°	.015884	.003601	−.007514	−.007479
	20°	.016675	.001879	.017388	−.007102
	25°	.017342	.000476	.049767	−.004227
	30°	.017631	.000000	.089673	.001803
$\lambda = 60$, $\mu = 1.0$	0°	.014523	.005613	−.039559	0
	5°	.014692	.005295	−.036086	−.003351
	10°	.015172	.004396	−.025617	−.006095
	15°	.015876	.003083	−.008017	−.007615
	20°	.016659	.001634	.016897	−.007281
	25°	.017322	.000439	.049285	−.004448
	30°	.017608	.000000	.089193	.001540
$\lambda = 100$, $\mu = 0.7$	0°	.008175	.006415	−.037538	0
	5°	.008340	.006052	−.034274	−.003181
	10°	.008807	.005016	−.024382	−.005789
	15°	.009494	.003503	−.007583	−.007236
	20°	.010261	.001933	.016495	−.006902
	25°	.010911	.000466	.048254	−.004133
	30°	.011193	.000000	.087927	.001751
$\lambda = 100$, $\mu = 1.0$	0°	.008183	.005441	−.038063	0
	5°	.008346	.005135	−.034794	−.003227
	10°	.008809	.004268	−.024889	−.005880
	15°	.009469	.002999	−.008076	−.007370
	20°	.010249	.001599	.016026	−.007077
	25°	.010895	.000430	.047799	−.004349
	30°	.011175	.000000	.087476	.001495

Table 4. Deflections, angles of twist, and bending and twisting moments for $2\gamma = 45^\circ$

	ϕ	$\frac{y}{V}$	$\frac{\beta}{V}$	$\frac{M_x}{aV}$	$\frac{M_z}{aV}$
	0°	.020635	.002941	-.031173	0
	5°	.020762	.002649	-.026338	-.002580
$\lambda = 60$	10°	.021102	.001865	-.011814	-.004315
$\mu = 0.7$	15°	.021541	.000864	.012440	-.004353
	20°	.021886	.000101	.046432	-.001860
	$22^\circ 30'$	.021943	.000000	.067053	.000607
	0°	.020634	.002463	-.031422	0
	5°	.020760	.002220	-.026586	-.002601
$\lambda = 60$	10°	.021099	.001569	-.012060	-.004358
$\mu = 1.0$	15°	.021535	.000734	.012196	-.004423
	20°	.021878	.000091	.046190	-.001946
	$22^\circ 30'$	.021935	.000000	.066811	.000510
	0°	.012143	.002912	-.030794	0
	5°	.012269	.002623	-.026048	-.002549
$\lambda = 100$	10°	.012605	.001848	-.011754	-.004269
$\mu = 0.7$	15°	.013039	.000857	.012230	-.004319
	20°	.013381	.000100	.046038	-.001848
	$22^\circ 30'$	.013438	.000000	.066633	.000601
	0°	.012144	.002438	-.031043	0
	5°	.012268	.002199	-.026296	-.002571
$\lambda = 100$	10°	.012603	.001555	-.011998	-.004312
$\mu = 1.0$	15°	.013035	.000728	.011999	-.004383
	20°	.013375	.000091	.045800	-.001933
	$22^\circ 30'$	.013432	.000000	.066395	.000506

Table 5. Deflections, angle of twist, and bending and twisting moments for $2\gamma = 30°$

	ϕ	$\frac{y}{V}$	$\frac{\beta}{V}$	$\frac{M_x}{aV}$	$\frac{M_z}{aV}$
$\lambda = 60$, $\mu = 0.7$	$0°$	.031661	.000894	-.021441	0
	$5°$	.031741	.000704	-.014127	-.001658
	$10°$	.031925	.000270	-.007792	-.002041
	$15°$	.032042	.000000	.044227	.000124
$\lambda = 60$, $\mu = 1.0$	$0°$	.031660	.000742	-.021521	0
	$5°$	.031740	.000585	-.014207	-.001665
	$10°$	.031923	.000225	.007712	-.002055
	$15°$	.032040	.000000	.044148	.000103
$\lambda = 100$, $\mu = 0.7$	$0°$	.018925	.000893	-.021390	0
	$5°$	.019006	.000703	-.014102	-.001655
	$10°$	.019189	.000269	.007765	-.002037
	$15°$	.019306	.000000	.044173	.000124
$\lambda = 100$, $\mu = 1.0$	$0°$	.018925	.000741	-.021470	0
	$5°$	.019005	.000584	-.014182	-.001662
	$10°$	.019188	.000225	.007686	-.002051
	$15°$	.019305	.000000	.044094	.000103

Table 6. Constrained Beam

φ (deg-min)	β (radians)	v (inches)	M_x (ft-tons)	M_z (ft-tons)
0°	$0.0^{3}1959$	0.4122	-58.391	0
5°	$0.0^{4}1764$	0.4148	-49.334	-4.833
10°	$0.0^{3}1242$	0.4216	-22.129	-8.082
15°	$0.0^{4}575$	0.4304	23.301	-8.165
20°	$0.0^{5}673$	0.4373	86.973	-3.484
$22^\circ30'$	0	0.4384	125.598	1.137

Table 7. Beam free to rotate

φ (deg-min)	β (radians)	v (Inches)	M_x (Ft.tons)	M_z (ft-tons)
0°	$0.0^{4}966$	0.4122	-61.268	0
5°	$0.0^{4}766$	0.4146	-52.222	-5.076
10°	$0.0^{4}220$	0.4216	-25.025	-8.579
15°	$0.0^{4}466$	0.4304	20.398	-8.916
20°	$0.0^{3}1035$	0.4372	84.084	-4.435
$22^\circ30'$	$0.0^{3}1106$	0.4384	122.708	0

Numerical Example

A reinforced concrete circular beam, having a radius $R = 25$ feet and a square cross-section of 30 inches by 30 inches, is acted upon by eight vertical loads of 75 tons each equally spaced at $2\gamma = 45^\circ$. The modulus of elasticity of concrete is assumed $E = 3(10)^6$ p.s.i. and the elastic constant of the foundation is supposed $k = 1500$ lbs./in./in.

It follows:

$$I_x = 6.75(10^4)\text{in}^4; \quad \lambda = 60, \ \mu = 0.7$$

$$\gamma = 0.0666; \ \alpha = 28{,}125 \text{ ft. Ton}, \ \alpha v = 1873.12 \text{ ft. Ton}$$

In Table 6 the results obtained for this particular case from Table 4 are given, while in Table 7 the results are given for a beam of the same characteristics and under the same loads, but free to rotate. The results given in Table 7 are taken from the Paper listed as No. [1] of the Bibliography.

If one compares the results given by the two last Tables, one sees that the values in the β columns are completely different, the values for v are practically the same, while the values in the M_x and M_z columns are fairly close.

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